Supersymmetries of Dirac Operators with Examples

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Supersymmetry in Integrable Models
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* continuation on space-time lattices with G. Bergner, T. Kaestner, S. Uhlmann, B. Wellegehausen, C. Wozar
Annals of Physics 323,...

Extended Supersymmetry of squared Dirac Operator

On the structure of the supercharges

Dirac-operator on complex projective spaces

From Diracoperator to susy lattice models

The supersymmetric Hydrogen atom

Extended Supersymmetry of $\not{D}^2$

- hermitean supercharges $\mathcal{Q}_i,\ i = 1, \dots, \mathcal{N}$

$$\delta_{ij} H = \frac{1}{2}\{\mathcal{Q}_i, \mathcal{Q}_j\} \Longrightarrow [\mathcal{Q}_i, H] = 0$$

- hermitean grading operator Γ

$$\{\mathcal{Q}_i, \Gamma\} = 0, \quad \Gamma^\dagger = \Gamma, \quad \Gamma^2 = \mathbb{1}$$

- $\text{spec}(\Gamma) = \pm 1$: Hilbert-space decomposes

$$\mathcal{H} = \mathcal{H}_{\rm B} \oplus \mathcal{H}_{\rm F}, \quad \mathcal{Q}_i : \mathcal{H}_{\rm B,F} \longrightarrow \mathcal{H}_{\rm F,B}$$

- prominent examples
  $d = 1$: Nicolai-Witten, $d > 1$: Andrianov, Borisov, Ioffe
  low-energy sector of susy field theories, ...

- two real supercharges

$$H = \mathcal{Q}_1^2 = \mathcal{Q}_2^2, \quad \{\mathcal{Q}_1, \mathcal{Q}_2\} = 0$$

$\Rightarrow$ nilpotent complex supercharge

$$\mathcal{Q} = \frac{1}{2}(\mathcal{Q}_1 + i\mathcal{Q}_2), \quad \mathcal{Q}^\dagger = \frac{1}{2}(\mathcal{Q}_1 - i\mathcal{Q}_2)$$

- Hamiltonian

$$H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}, \quad \mathcal{Q}^2 = 0 \quad \text{and} \quad [\mathcal{Q}, H] = 0$$

- four real supercharges $\Rightarrow$ 2 nilpotent supercharges ...
- realizations: Euclidean Dirac operator in curved spaces

$$G_{MN} = E_M^A E_N^B \delta_{AB} \quad , \quad \{\Gamma^M, \Gamma^N\} = 2G^{MN}$$

- covariant derivative on spinors (with gauge field A_M)

$$D_M = \partial_M + \Omega_M + A_M$$

  geometry: spin-connection Ω_M
- hermitean Dirac operator in D dimensions

$$i\not{D} = i\Gamma^M D_M$$

- even dimensions: generalization Γ of γ_5 with $\{\Gamma, \not{D}\} = 0$
- 'trivial' chiral $\mathcal{N} = 2$ supersymmetry

$$\mathcal{Q}_1 = i\not{D}, \quad \mathcal{Q}_2 = \Gamma\not{D}$$

- here: aiming at finer complex structure

- super-Hamiltonian

$$H = -\not{D}^2 = -G^{MN} D_M D_N - \frac{1}{2}\Gamma^{MN}\mathcal{F}_{MN}$$

- $\mathcal{F}_{MN}$: Riemann-curvature and Yang-Mills field strength

$$\mathcal{F}_{MN} = [D_M, D_N] = F_{MN} + R_{MN}$$

- question: are there other first order operators $\mathcal{Q}_I$ with $\mathcal{Q}_I^2 = -\not{D}^2$ and forming a super-algebra?

- Class of operators: free Dirac operator, 2 dimensions, Rittenberg + deCrombrugghe (1983), our earlier work

$$\mathcal{Q}(\mathcal{I}) = i\mathcal{I}^M{}_N \Gamma^N D_M, \quad \mathcal{I}^M{}_N(x): \quad \text{real tensor field}$$

Lemma: *The $\mathcal{N}$ hermitean charges*

$$\mathcal{Q}(\mathbb{1}) = i\not{D} \quad \textit{and} \quad \mathcal{Q}(\mathcal{I}_1), \ldots, \mathcal{Q}(\mathcal{I}_{\mathcal{N}-1})$$

generate an extended superalgebra $\Longleftrightarrow \mathcal{I}_i^T = -\mathcal{I}_i$ *and*

$$\{\mathcal{I}_i, \mathcal{I}_j\} = -2\delta_{ij}\mathbb{1}_D, \quad \nabla\mathcal{I}_i = 0, \quad [\mathcal{I}_i, F] = 0$$

- $\mathcal{I}_i$ define complex structures
- integrability condition for $\nabla\mathcal{I} = 0 \Longrightarrow [\mathcal{I}, R] = 0$
- $\mathcal{N} = 2 \Longleftrightarrow$ space is Kähler and $[\mathcal{I}, F] = 0$
- $\mathcal{N} = 4 \Longleftrightarrow$ space is hyper-Kähler and $[\mathcal{I}_i, F] = 0$
- flat space with $D = 4 \Rightarrow F$ (anti) selfdual

- $\mathcal{N} = 2 \Rightarrow$ even-dimensional space $D = 2n$:

$$\mathcal{I} = i\sigma_2 \otimes \mathbb{1}_n \Longrightarrow F = \sigma_0 \otimes A + i\sigma_2 \otimes S$$
$$A: \text{ anti-symmetric}, \quad S: \text{ symmetric}$$

- $D = 4$ dimensions: $E_1 = B_1$ and $E_3 = B_3$
- $\mathcal{N} = 4 \Rightarrow$ dimension $D = 4n : \mathcal{I}_i = \tau_i \otimes \mathbb{1}_n$

$$\text{ASD}: \ \{\tau_1, \tau_1, \tau_3\} = \{i\sigma_0 \otimes \sigma_2, i\sigma_2 \otimes \sigma_3, i\sigma_2 \otimes \sigma_1\}$$

- field strength

$$F = \mathbb{1}_4 \otimes A + \tilde{\tau}_1 \otimes S_1 + \tilde{\tau}_2 \otimes S_2 + \tilde{\tau}_3 \otimes S_3$$
$$\text{SD}: \ \{\tilde{\tau}_1, \tilde{\tau}_1, \tilde{\tau}_3\} = \{i\sigma_3 \otimes \sigma_2, i\sigma_2 \otimes \sigma_0, i\sigma_1 \otimes \sigma_2\}$$

- 4 dimensions: F self-dual: $\boldsymbol{E} = \boldsymbol{B}$ (Annals of Physics 315)

On the structure of the supercharges

- local coordinates $\{z^\mu, \bar{z}^{\bar{\mu}}\}$ on complex manifold
- change of coordinates $x^M \leftrightarrow \{z^\mu, \bar{z}^{\bar{\mu}}\},\ \mu = 1, \dots, n$

$$dz^\mu = f^\mu{}_M dx^M \quad , \quad d\bar{z}^{\bar{\mu}} = f^{\bar{\mu}}{}_M dx^M$$
$$\partial_\mu = f^M{}_\mu \partial_M \quad , \quad \partial_{\bar{\mu}} = f^M{}_{\bar{\mu}} \partial_M$$

- complex structure

$$i\mathcal{I}^M{}_N = f^M{}_\mu f^\mu{}_N - f^M{}_{\bar{\mu}} f^{\bar{\mu}}{}_N$$

- line element

$$ds^2 = G_{MN} dx^M dx^N = 2h_{\mu\bar{\nu}} dz^\mu d\bar{z}^{\bar{\nu}}$$

- Kähler space: $h_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} K$

- complex covariant derivative

$$D_\mu = f^M{}_\mu D_M = \partial_\mu + \omega_\mu + A_\mu$$
$$D_{\bar\mu} = f^M{}_{\bar\mu} D_M = \partial_{\bar\mu} + \omega_{\bar\mu} + A_{\bar\mu}$$

- lowering/raising operator

$$\psi^\mu = \frac{1}{2} f^\mu{}_M \Gamma^M \quad , \quad \psi^{\dagger\bar\mu} = \frac{1}{2} f^{\bar\mu}{}_M \Gamma^M \Rightarrow$$
$$\{\psi^\mu, \psi^\nu\} = 0 \quad , \quad \{\psi^\mu, \psi^{\dagger\bar\nu}\} = \frac{1}{2} h^{\mu\bar\nu},$$

- conserved fermion-number operator

$$N = 2h_{\bar\mu\nu}\psi^{\dagger\bar\mu}\psi^\nu \Longrightarrow [N, \psi^\sigma] = -\psi^\sigma$$

- use $\delta^M{}_N = f^M{}_\mu f^\mu{}_N + f^M{}_{\bar\mu} f^{\bar\mu}{}_N \Longrightarrow$ decomposition

$$i\not{D} = \mathcal{Q} + \mathcal{Q}^\dagger \equiv 2i\psi^\mu D_\mu + 2i\psi^{\dagger\bar\mu} D_{\bar\mu}$$

- Kähler and $F=\mathcal{I}^T F\mathcal{I} \Longrightarrow$

$$[D_\mu, D_\nu]=\mathcal{F}_{\mu\nu}=f^M{}_\mu f^N{}_\nu \mathcal{F}_{MN}=0$$

- integrability condition for complex superpotential

$$\omega_\mu + A_\mu = g\left(\partial_\mu g^{-1}\right)$$

- $\omega_\mu,\ A_\mu\ \in$ complexified Lie algebras
- $\Rightarrow$ very useful deformation formula

$$\mathcal{Q}=g\mathcal{Q}_0 g^{-1}\quad,\quad \mathcal{Q}^\dagger = g^{-1\dagger}\mathcal{Q}_0^\dagger g^\dagger$$
$$\mathcal{Q}_0=\psi_0^\mu\partial_\mu\quad,\quad \mathcal{Q}_0^\dagger=\psi_0^{\dagger\bar\mu}\partial_{\bar\mu}$$

- constant fermionic lowering and raising operators

$$\psi_0^\mu = g^{-1}\psi^\mu g\quad,\quad \psi_0^{\dagger\bar\mu}=g^\dagger\psi^\mu g^{-1\dagger}$$

- problem: how to calculate prepotential g?

$$g = g_A g_\omega \qquad g_A = \text{path ordered integral of } A_\mu$$
$$g_\omega = \text{complex } n\text{-bein in spin-representation}$$

- Summary
  - if $i\not{D}$ admits extended complex supersymmetry $\Rightarrow$

$$i\not{D} = \mathcal{Q} + \mathcal{Q}^\dagger, \quad \mathcal{Q}^2 = \mathcal{Q}^{\dagger 2} = 0, \quad \mathcal{Q} = g^{-1}\mathcal{Q}_0 g$$

  - $H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}$ commutes with $N \Rightarrow$ decomposition of $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n \quad \text{with} \quad N|_{\mathcal{H}_p} = p \cdot \mathbb{1}$$

  - block-diagonal form of super-Hamiltonian

$$H\big|_{\mathcal{H}_p} \qquad \text{second order matrix-Differential operator}$$

  - complex $\mathcal{Q}^\dagger : \mathcal{H}_p \to \mathcal{H}_{p+1}$, $\mathcal{Q} : \mathcal{H}_p \to \mathcal{H}_{p-1}$

Dirac-operator on $\mathbb{C}P^n$

- homogeneous coordinates $u \in \mathbb{C}^{n+1}$, $\bar{u} \cdot u = 1$
- local coordinates

$$u = \frac{1}{\rho}(1, z), \quad \rho^2 = 1 + \bar{z} \cdot z, \quad z \in \mathbb{C}^n$$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{\rho^2} - \frac{(\bar{z} \cdot dz)(z \cdot d\bar{z})}{\rho^4} \Longrightarrow h_{\bar{\mu}\nu}$$

- Kähler potential $K = \log \rho^2$

$$h_{\bar{\mu}\nu} = \partial_{\bar{\mu}} \partial_\nu K = \frac{1}{2} \delta_{\bar{\alpha}\beta} e^{\bar{\alpha}}_{\bar{\mu}} e^{\beta}_{\nu}$$

- connection (1,0)-Form $\omega^{\alpha}_{\mu\beta} = e^{\beta\bar{\sigma}} \partial_\mu e_{\bar{\sigma}\alpha}$, etc.

- n even: must add $U(1)$ gauge potential

$$A = \frac{k}{4}(\partial - \bar{\partial})K \Longrightarrow g_A = e^{-kK/4} = \rho^{-k/2}$$

- n even (odd) $\Rightarrow$ k odd (even)
- pre-potential g_ω known, complicated

• Zero-modes of $\not{D}$

- index theorem on $\mathbb{C}P^n$ — Dolan 2002

$$\text{index}(i\not{D}) = \frac{(q+1)(q+2)\dots(q+n)}{n!}, \quad q = \frac{k-n-1}{2}$$

- all zero-modes are in extremal sector $N = n$

$$g_\omega\big|_{N=n} = \rho^{\frac{n+1}{2}}$$

- let $\chi\in\mathcal{H}_n\Longrightarrow\mathcal{Q}^\dagger\chi=0$ algebraically
- remains

$$0=\mathcal{Q}\chi\Longleftrightarrow\mathcal{Q}_0\left(g^{-1}\chi\right)=0,\quad g=\rho^{(n+1-k)/2}$$

- all explicit zero-modes (Ivanov, Mezincescu, Townsend 2004)

$$\chi=g(\rho)\,(\bar z^{\bar 1})^{m_1}\cdots(\bar z^{\bar n})^{m_n}\,\psi^{\dagger\bar 1}\cdots\psi^{\dagger\bar n}|0\rangle$$

- square integrable for

$$\sum_{i=1}^{n}m_i\in\{0,1,2,\dots,q\}$$

- total number = index($\not{D}$)

From $\not{D}$ to susy-QM in n dimensions

- $2n$-dimensional flat space, $U(1)$ potential A_M
- $[F, \mathcal{I}] = 0 \Longrightarrow \mathcal{N} = 2$ susy, conserved N
- if $A^M \partial_M$ vanishes $\Longrightarrow \not{D}^2$ matrix-Schrödinger-operator
- dimensional torus reduction

$$\begin{aligned} \text{space} \quad &: \quad \mathbb{R} \times \ldots \times \mathbb{R} \times S^1 \times \ldots \times S^1 \\ \text{coordinates} \quad &: \quad (x^1, \ldots, x^n; \theta^1, \ldots, \theta^n), \; z^a = x^a + i\theta^a \end{aligned}$$

- $A_M = A_M(x^1, \ldots, x^n) \Longrightarrow$ set $\partial_{\theta^a} = 0 \Longleftrightarrow \partial_{z^a} = \frac{1}{2}\partial_{x^a}$
- assume further $A_1 = A_2 = \cdots = A_n = 0 \Longrightarrow$

$$A_a = g \frac{\partial}{\partial z^a} g^{-1} = \frac{1}{2} e^{-\chi(x)} \frac{\partial}{\partial x^a} e^{\chi(x)}, \quad \chi(x) \in \mathbb{R}$$

- nilpotent supercharges

$$\begin{aligned} \mathcal{Q} &= e^{-\chi}\mathcal{Q}_0 e^{\chi}, \quad \mathcal{Q}_0 = i\psi^a\partial_a \\ \mathcal{Q}^\dagger &= e^{\chi}\mathcal{Q}_0^\dagger e^{-\chi}, \quad \mathcal{Q}_0^\dagger = i\psi^{a\dagger}\partial_a \end{aligned}$$

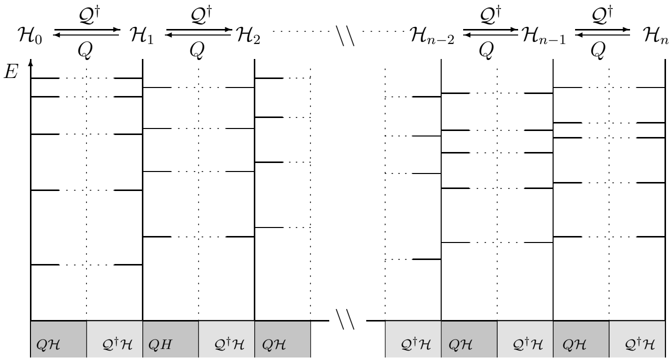
- de- and increase conserved fermion number by 1

$$N = \sum_{a=1}^{n} \psi_a^\dagger \psi_a$$

- N-conserving super-Hamiltonian ($\chi_{ab} = \partial_a\partial_b\chi$)

$$H = \left(-\triangle + (\nabla\chi)^2 + \triangle\chi\right)\mathbb{1}_{2^d} - 2\sum_{a,b=1}^{d} \psi_a^\dagger\, \chi_{ab}\, \psi_b$$

Andrianov, Borisov, Ioffe (1984); Cooper, Khare, Musto, Wipf (1988)



- decomposition

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$$

- $H|_{\mathcal{H}_p}$ matrix Schrödinger operator, $\binom{n}{p}$- dim. matrix

$\mathcal{N}=2$ Susy lattice field theories

- 1-dimensional lattice, sites $n\in\{1,\dots,N\}$
- lattice field $\phi(n)\in\mathbb{R}^2$ and momentum field $\pi(n)$
- need $2N$ variables x^a (Dirac operator in $4N$ dimensions)
- identification: coordinates $\leftrightarrow$ lattice fields

$$\phi(n)=\begin{pmatrix}x^{2n-1}\\ x^{2n}\end{pmatrix}\quad \psi(n)=\begin{pmatrix}\psi^{2n-1}\\ \psi^{2n}\end{pmatrix}\quad \psi^\dagger(n)=\begin{pmatrix}\psi^{\dagger\,2n-1}\\ \psi^{\dagger 2n}\end{pmatrix}$$

- free supercharges

$$\mathcal{Q}_0=i\sum_{\substack{n=1\\ a=1,2}}^{n=N}\psi_a(n)\frac{\partial}{\partial\phi_a(n)}\quad,\quad \mathcal{Q}_0^\dagger=i\sum_{\substack{n=1\\ a=1,2}}^{n=N}\psi_a^\dagger(n)\frac{\partial}{\partial\phi_a(n)}$$

- how to choose $\chi(x)\equiv\chi(\phi)$?

- Dirac-Hamiltonian in 2 dimensions

$$\int \psi^\dagger h_F \psi \quad \text{with} \quad h_F = -i\gamma_*\partial + m\gamma^0$$

- comes from $\sum \psi^\dagger \chi'' \psi \Longrightarrow \chi \propto \sum \phi h_F \phi + \dots$
- χ real $\Longrightarrow$ choose adapted representation

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_1, \quad \gamma_* = \gamma^0\gamma^1 = -\sigma_2$$

- free massive field theory

$$\chi^m = -\tfrac{1}{2}(\phi, h_F\phi) = -\tfrac{1}{2}(\phi, h_F^0\phi) + \sum_n f\big(\phi(n)\big)$$

- with harmonic target-space function

$$f(\phi) = -\tfrac{1}{2}m(\phi, \gamma^0\phi) = \tfrac{1}{2}m(\phi_2^2 - \phi_1^2)$$

- interacting theory

$$\chi = -\tfrac{1}{2}(\phi, h_F^0\phi) + \sum f\big(\phi(n)\big), \qquad \Delta f = 0$$

- let $f(\phi) + ig(\phi)$ be analytic function of $\phi_1 + i\phi_2$
- bosonic part of super-Hamiltonian

$$H_B = \tfrac{1}{2}(\pi,\pi) - \tfrac{1}{2}(\phi, \triangle\phi) + \tfrac{1}{2}\left(\frac{\partial f}{\partial\phi}, \frac{\partial f}{\partial\phi}\right) + \mathcal{Z}$$

- would-be central charge

$$\mathcal{Z} = \left(\frac{\partial g}{\partial\phi_1}, \partial^\dagger\phi_1\right) - \left(\frac{\partial g}{\partial\phi_2}, \partial\phi_2\right)$$

- fermionic part with Yukawa-term

$$H_F = (\psi, h_F^0\psi) - \big(\psi, \gamma^0\Gamma_\phi\psi\big)\,, \quad \Gamma_\phi = f_{,11}(\phi) - i\gamma_* f_{,12}(\phi)$$

- example: cubic superpotential $\Rightarrow \phi^4$ model

$$f + ig = \lambda(\phi_1 + i\phi_2)^3/3$$

- super-Hamiltonian

$$\begin{aligned} H_B &= \tfrac{1}{2}(\pi,\pi) - \tfrac{1}{2}(\phi,\Delta\phi) + (\psi, h_F^0\psi) + \tfrac{1}{2}\lambda^2\,(\phi,\phi)^2 + \mathcal{Z} \\ H_F &= -2\lambda\left(\psi,\gamma^0(\phi_1 + i\gamma_*\phi_2)\psi\right) \end{aligned}$$

- 'central term' = almost surface term (no Leibniz-rule)

$$\mathcal{Z} = 2\lambda\left(\phi_1\phi_2, \partial^\dagger\phi_1\right) - \lambda\left(\phi_1^2 - \phi_2^2, \partial\phi_2\right)$$

- ground state for quadratic f known: in sector $\mathcal{H}_N$ of

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{2N-1} \oplus \mathcal{H}_{2N}$$

- by construction: lattice models have partial susy
- non-standard action: choice of ∂ is important!
- ground states in strong coupling limit known
- number for arbitrary coupling known
- similar construction and results for $\mathcal{N} = 1$ model
- dimensional reduction of $\not{D}$ on curved spaces $\Rightarrow$ supersymmetric lattice sigma-models

  Kirchberg, Länge, Wipf, Annals of Physics 316, 357
- generalisatons?
- starting point for high precision simulations of WZW

  Kästner, Bergner, Uhlmann, Wipf, Wozar: Phys. Rev. D 78, page 095001

The supersymmetric Hydrogen atom

- motion in Newton/Coulomb potential

  Hermann, Bernoulli, Laplace, Runge, Lenz, Pauli

- angular momentum, Runge-Lenz vector

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \quad , \quad \boldsymbol{C} = \frac{1}{2m}(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}) - \frac{e^2}{r}\,\boldsymbol{r}$$

- on bound states

$$\boldsymbol{K} = \sqrt{\frac{-m}{2H}}\,\boldsymbol{C}$$

- dynamical $SO(4)$ symmetry ($\hbar = 1$)

$$\begin{aligned} [L_a, L_b] &= i\epsilon_{abc}L_c \\ [L_a, K_b] &= i\epsilon_{abc}K_c \\ [K_a, K_b] &= i\epsilon_{abc}L_c \end{aligned}$$

- Casimirs

$$\mathcal{C} = \boldsymbol{L}^2 + \boldsymbol{K}^2 \quad , \quad \mathcal{C}' = \boldsymbol{L}\cdot\boldsymbol{K} = 0$$

- Coulomb-Hamiltonian

$$H = -\frac{me^4}{2}\frac{1}{\mathcal{C}+\hbar^2}$$

- only symmetric representations
- group theory $\Rightarrow$ spectrum, wave functions
- d dimensions: dynamical $SO(d+1)$-symmetry

$$L_{ab}, \quad K_a \propto C_a = L_{ab}p_b + p_b L_{ab} - \eta x_a/r$$

supersymmetric Hydrogen atom ($d = 3$)

- dimensional reduction of 6-dimensional $\not{D}$
- in 3 dimensions: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and

$$\begin{aligned} H = \{\mathcal{Q}, \mathcal{Q}^\dagger\} &= H_0 \otimes \mathbb{1}_{2^d} - 2 \sum \psi_a^\dagger \psi_b \partial_a \partial_b \chi \\ &= H_3 \otimes \mathbb{1}_{2^d} + 2 \sum \psi_a \psi_b^\dagger \partial_a \partial_b \chi \end{aligned}$$

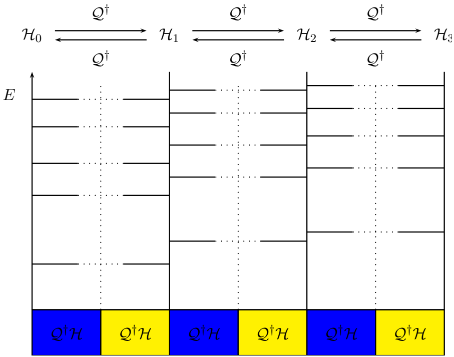
- H commutes with $\mathcal{Q}$, $\mathcal{Q}^\dagger$ and N, $\mathcal{Q}^2 = 0$
- $H|_{\mathcal{H}_0},\ H|_{\mathcal{H}_3}$ ordinary Schrödinger operators

$$\begin{aligned} H_0 &= -\triangle + (\nabla\chi, \nabla\chi) + \triangle\chi \\ H_3 &= -\triangle + (\nabla\chi, \nabla\chi) - \triangle\chi \end{aligned}$$

- $H|_{\mathcal{H}_1}, H|_{\mathcal{H}_2}$ are 3×3-matrix-Schrödinger operators

- Hodge-decomposition

$$\mathcal{H} = \mathcal{Q}\mathcal{H} \oplus \mathcal{Q}^\dagger\mathcal{H} \oplus \text{Ker}\, H$$



- particular superpotential $\chi(r) = -\lambda r$

$$H = \left(-\triangle + \lambda^2\right) \mathbb{1}_8 - \frac{2\lambda}{r} B, \quad B = \mathbb{1} - N + \mathcal{S}^\dagger \mathcal{S}$$

- lowering operator $\mathcal{S} = \hat{\boldsymbol{x}} \cdot \boldsymbol{\psi}$
- Hamiltonians in sectors $N = 0$ and $N = 3$

$$H_0 = -\triangle + \lambda^2 - \frac{2\lambda}{r} \qquad pe^- \text{ system}$$

$$H_3 = -\triangle + \lambda^2 + \frac{2\lambda}{r} \qquad pe^+ \text{ system}$$

- susy extension of conserved total angular momentum

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{x} \times \boldsymbol{p} - i\boldsymbol{\psi}^\dagger \times \boldsymbol{\psi}$$

- $\boldsymbol{x}, \boldsymbol{\psi}$ vectors; $\mathcal{S}, B$ scalars

- susy extension of conserved Runge-Lenz-vector

$$\boldsymbol{C} = \boldsymbol{p} \wedge \boldsymbol{J} - \boldsymbol{J} \wedge \boldsymbol{p} - 2\lambda\,\hat{\boldsymbol{x}} B$$

- discrete spectrum $\subset [0, \lambda^2)$:

$$\boldsymbol{K} = \frac{1}{2}\frac{\boldsymbol{C}}{\sqrt{\lambda^2 - H}}$$

- $\boldsymbol{J}, \boldsymbol{K}$ generate $SO(4)$ symmetry algebra
- no algebraic relation $H = H(N, \boldsymbol{J}, \boldsymbol{K})$, but

$$\begin{aligned}\lambda^2\mathcal{C} = \boldsymbol{K}^2 H \quad &+ \quad \left(\boldsymbol{J}^2 + (1-N)^2\right)\mathcal{Q}\mathcal{Q}^\dagger \\ &+ \quad \left(\boldsymbol{J}^2 + (2-N)^2\right)\mathcal{Q}^\dagger\mathcal{Q}\end{aligned}$$

- second order Casimir

$$\mathcal{C} = \boldsymbol{J}^2 + \boldsymbol{K}^2$$

- $\mathcal{Q}\mathcal{H},\ \mathcal{Q}^\dagger\mathcal{H}$, Ker($H$) invariant under H

$$H\big|_{\mathcal{Q}\mathcal{H}} = \lambda^2\,\frac{\mathcal{C}}{(1-N)^2+\mathcal{C}}$$
$$H\big|_{\mathcal{Q}^\dagger\mathcal{H}} = \lambda^2\,\frac{\mathcal{C}}{(2-N)^2+\mathcal{C}}$$

- supersymmetric ground state: $SO(4)$ singlet
- realization of $so(4)$ on $\mathcal{H} \Longrightarrow$ allowed representations
- discrete spectrum, degeneracies, eigenfunctions
- generalization to higher dimensions $\Longrightarrow$
  branching rules $SO(d-1) \longrightarrow SO(d)$
  $\Longrightarrow$ allowed $SO(d+1)$ representations
  Kirchberg, Länge, Pisani, Wipf, Annals of Physics 303, page 359

- hyper-Kähler:
  $4k$-dimensional manifold with holonomy $Sp(k) \Rightarrow$
  Ricci flat, Calabi-Yau, since $Sp(k) \subset SU(2k)$
  $d = 4$: K_3-manifold, T^4
  $k > 4$: Kummer varieties, ...
  non-compact: G/H, G quaternions, $H \subset Sp(1)$ discrete
  dimensional reduction of sd-Yang-Mills
  instanton and monopole moduli spaces
- Kähler:
  $\mathbb{C}^n$, $\mathbb{C}^n/\Lambda$, Riemann-surfaces, $\mathbb{C}P^n$, K_3