CRYSTALS, INSTANTONS AND QUANTUM GEOMETRY

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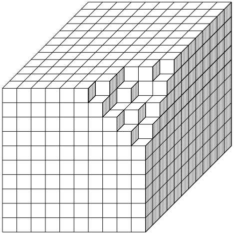
Supersymmetry in Integrable Systems
Yerevan, Armenia 2010

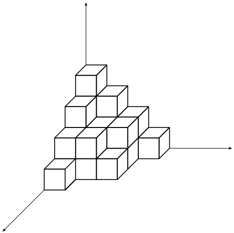
Melting crystal model in 3D

(Okounkov, Reshetikhin & Vafa '06)



- Unit cube at $(I, J, K) \in \mathbb{Z}^3_{\geq 0} \subset \mathbb{R}^3$ evaporates $\iff$ all $(i \leq I, j \leq J, k \leq K)$ already evaporated
- Removing each atom from corner of crystal contributes $q = \mathrm{e}^{-\mu/T}$ to Boltzmann weight

Plane partitions = 3D Young diagrams



Piling $\pi_{i,j}$ cubes vertically at position $(i, j, 0)$ gives rectangular array:

$$\pi = (\pi_{i,j}) \quad \text{such that } \pi_{i,j} \geq \pi_{i+1,j} \ , \ \pi_{i,j} \geq \pi_{i,j+1}$$

[Recall: ordinary partition = Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_i \geq \lambda_{i+1} \geq 0$, λ_i = length of i-th row]

Statistical mechanics of crystal melting

Canonical ensemble in which each π has energy $\propto |\pi| = \sum_{i,j\geq 1} \pi_{i,j} =$ total number of cubes:

$$
\begin{aligned}
Z_{\mathbb{C}^3} &:= \sum_{\pi} q^{|\pi|} \\
&= \sum_{N=0}^{\infty} pp(N)\, q^N \\
&= \prod_{n=1}^{\infty} \frac{1}{\left(1-q^n\right)^n} =: M(q) \qquad \text{(MacMahon function)}
\end{aligned}
$$

$pp(N) =$ number of plane partitions π with $|\pi| = N$

Generalizations — Calabi–Yau crystals

Trivalent planar graph Γ with:

(1) $3D$ partition π_v at each vertex v

(2) $2D$ partition λ_e at each edge e (asymptotics of π_v)

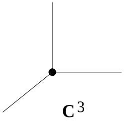
"Topological string" partition function on CY_3 X with toric diagram Γ (Aganagic *et al.* '05; Maulik *et al.* '06):

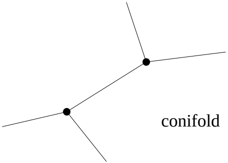
$$Z_X = \sum_{\substack{\text{Young tableaux} \\ \lambda_e}} \prod_{\text{edges } e} Q_e^{|\lambda_e|} \prod_{\substack{\text{vertices} \\ v=(e_1,e_2,e_3)}} M_{\lambda_{e_1},\lambda_{e_2},\lambda_{e_3}}(q)$$

Generating function for plane partitions π with boundaries λ, μ, ν:

$$M_{\lambda,\mu,\nu}(q) = \sum_{\pi\,:\,\partial\pi=(\lambda,\mu,\nu)} q^{|\pi|}$$

Example — Conifold





$$Z_{\text{conifold}} = \sum_{\lambda} M_{\emptyset,\emptyset,\lambda}(q)\, M_{\emptyset,\emptyset,\lambda}(q)\, Q^{|\lambda|}$$

$$= \sum_{\pi_v} q^{|\pi_v| + \sum_{(i,j)\in\lambda}(i+j+1)}\, Q^{|\lambda|} = M(q)^2\, M(Q,q)^{-1}$$

$$M(Q,q) = \prod_{n=1}^{\infty} \frac{1}{\left(1 - Q\,q^n\right)^n}$$ counts weighted plane partitions

Free fermion representation

(Nakatsu & Takasaki '09; Sulkowski '09)

- Complex fermion field:

$$\psi(z) = \sum_{m\in\mathbb{Z}} \psi_m \, z^{-m-1} \ , \qquad \{\psi_m, \psi_n^*\} = \delta_{m+n,0}$$

- Fock space spanned by states labelled by Young tableaux; use modes α_n of bosonized field $\phi = :\psi(z)\,\psi^*(z):$ to define vertex operators:

$$\Gamma_\pm(x) = \exp\Big(\sum_{n>0} \frac{x^n}{n}\,\alpha_{\pm\,n}\Big) \ , \qquad [\alpha_m, \alpha_n] = m\,\delta_{m+n,0}$$

- Gives fermionic representation:

$$Z_{\mathbb{C}^3} = \langle 0|\cdots\Gamma_+(q^2)\,\Gamma_+(q)\,\Gamma_+(1)\,\Gamma_-(1)\,\Gamma_-(q)\,\Gamma_-(q^2)\cdots|0\rangle$$

- Identifies Z_X as τ-function of 1D Toda hierarchy

Unitary one-matrix models

(Ooguri, Sulkowski & Yamazaki '10; RS & Tierz '10)

$$Z_{\mathbb{C}^3} \;=\; \int_{U(\infty)} \mathrm{d}U \;\det \Theta(U|q)$$

$$Z_{\text{conifold}} \;=\; \int_{U(\infty)} \mathrm{d}U \;\det\Big(\frac{\Theta(U|q)}{\Theta(Q\,U|q)} \;\prod_{n=1}^{\infty} \big(1 + Q^{-1}\,U^{-1}\,q^n\big)\Big)$$

$$\Theta(u|q) \;=\; \sum_{j=-\infty}^{\infty} q^{j^2/2}\,u^j$$

Proof: Express $M_{\lambda,\mu,\nu}(q)$ as sum over **all** Young diagrams λ of "skew Schur functions", use Gessel's theorem to write as Toeplitz determinant

Chern–Simons gauge theory

- Chern–Simons theory on 3-manifold M with gauge group $U(N)$:

$$Z^N_{\rm CS}(M) = \int {\rm D}A \ {\rm e}^{\,{\rm i}\,S_{\rm CS}[A]}$$

$$S_{\rm CS}[A] = \frac{k}{4\pi} \int_M \ {\rm Tr}\left(A \wedge {\rm d}A + \tfrac{2}{3}\, A \wedge A \wedge A\right)$$

- On $M = S^3$ related to N-particle Sutherland model (RS & Tierz '10)
- By means of Hopf fibration $S^3 \longrightarrow S^2$ equivalent to "q-deformed" Yang–Mills theory on S^2; generalizes to other Seifert 3-manifolds $M \longrightarrow \Sigma$

  (Beasely & Witten '05; Caporaso *et al.* '06; Blau & Thompson '06; Griguolo *et al.* '07)

Finite N crystal model = Chern–Simons matrix model

- On $M = S^3$ equivalent to Stieltjes–Wigert matrix model (Mariño '04; Aganagic *et al.* '04; Tierz '04):

$$Z_{\rm CS}^N(S^3) = \int_{u(N)} \mathrm{d}H \ \mathrm{e}^{-\,\mathrm{Tr}\,\log^2 H/2g_s} = \prod_{j=1}^{N-1} \left(1-q^j\right)^{N-j}$$

$$q = \mathrm{e}^{-g_s} = \mathrm{e}^{-2\pi\,\mathrm{i}\,/(k+N)}$$

- Undetermined moment problem also described by unitary matrix model (Okuda '05):

$$Z_{\rm CS}^N(S^3) = \int_{U(N)} \mathrm{d}U \ \det \Theta(U|q)$$

- Hence: $Z_{\mathbb{C}^3} = \lim_{N\to\infty} Z_{\rm CS}^N(S^3)$

Kähler quantum gravity

(Iqbal *et al.* '06)

- X = complex manifold, $\dim_{\mathbb{C}}(X) = 3$, with nondegenerate Kähler (1,1)-form ω, $\mathrm{d}\omega = 0$ (usually toric CY_3)

- Gravitational path integral:

$$Z_X = \sum_{\substack{\text{quantized} \\ \omega}} \mathrm{e}^{-S} , \qquad S = \frac{1}{g_s^2} \int_X \frac{1}{3!} \omega \wedge \omega \wedge \omega$$

- Decompose "macroscopic" ω into "background" ω_0 and curvature F_A of holomorphic line bundle over X:

$$\omega = \omega_0 + g_s F_A , \qquad \int_\beta F_A = 0 \quad \forall \beta \in H_2(X, \mathbb{Z})$$

Kähler quantum gravity

- Gives action:

$$S = \frac{1}{g_s^2}\frac{1}{3!}\int_X \omega_0^3 + \frac{1}{2}\int_X F_A \wedge F_A \wedge \omega_0 + g_s \int_X \frac{1}{3!} F_A \wedge F_A \wedge F_A$$

- Statistical sum:

$$Z_X = \sum_{\substack{\text{line}\\ \text{bundles}}} q^{\mathrm{ch}_3} \prod_{i=1}^{b_2(X)} \left(Q_i\right)^{\int_{C_i} \mathrm{ch}_2}$$

$$q = \mathrm{e}^{-g_s}, \quad Q_i = \mathrm{e}^{-\int_{S_i} \omega_0}, \quad S_i \in H_2(X,\mathbb{Z}), \quad C_i \in H_4(X,\mathbb{Z})$$

- Problem: Fluctuation condition on F_A implies $\mathrm{ch}_2 = \mathrm{ch}_3 = 0$!

Quantization of geometry

- Take F_A to correspond to **singular** $U(1)$ gauge field A on X
- Instanton solutions of gauge theory on noncommutative deformation $\mathbb{C}^3_\theta$ described in terms of **ideals** $\mathcal{I}$ in polynomial ring $\mathbb{C}[z^1, z^2, z^3]$; correspond locally to crystalline configurations on each patch of X
- Become non-singular on blow-up:

$$X \longrightarrow \widehat{X} \qquad \text{(Quantum Foam)}$$

(Iqbal *et al.* '06)

- Hence molten crystal gives discretization of geometry of X at Planck scale;
  each atom of crystal is a fundamental unit of the geometry

6D cohomological gauge theory — Instantons

(Iqbal *et al.* '06; Cirafici, Sinkovics & RS '09)

- Topological twist of maximally SUSY–YM in $6D$

  $\Longleftrightarrow$ dimensional reduction of SYM in $10D$ on X:

$$S_{\rm bos} = \frac{1}{2}\int_X \Big({\rm d}_A\Phi\wedge *{\rm d}_A\overline{\Phi} + \big|F_A^{2,0}\big|^2 + \big|F_A^{1,1}\big|^2 \Big) + \frac{1}{2}\int_X \Big(F_A\wedge F_A\wedge\omega_0 + \frac{g_s}{3}\, F_A\wedge F_A\wedge F_A \Big)$$

- Gauge theory localizes at BRST fixed points:

$$F_A^{2,0} = 0 = F_A^{0,2}\ , \qquad F_A^{1,1}\wedge\omega_0\wedge\omega_0 = 0$$

- **Donaldson–Uhlenbeck–Yau equations:**

  BPS solutions $\equiv$ (generalized) instantons

6D cohomological gauge theory — Localization

- Localization onto instanton moduli space $\mathcal{M}$:

$$Z_X = \int_{\mathcal{M}} e(\mathcal{N})$$

$e(\mathcal{N})$ = Euler characteristic class of antighost bundle $\mathcal{N}$

- Regularize IR singularities on $\mathcal{M}$ for $X = \mathbb{C}^3$ by putting gauge theory in "Ω-background" (Nekrasov '04);
Since $\mathrm{ch}_2 = 0$, saturates Z_X by pointlike instantons

- Resolve small instanton UV singularities of $\mathcal{M}$ on $X = \mathbb{C}^3 \cong \mathbb{R}^6 \longrightarrow \mathbb{R}^6_\theta$:

$$\left[x^i , x^j\right] = \mathrm{i}\,\theta^{ij}$$

Noncommutative gauge theory

- Represent $z^a = x^{2a-1} - \mathrm{i}\, x^{2a}$, $\bar{z}^{\bar{a}} = x^{2a-1} + \mathrm{i}\, x^{2a}$, on Fock space:

$$\mathcal{H} = \mathbb{C}\big[\bar{z}^{\bar{1}}\,, \bar{z}^{\bar{2}}\,, \bar{z}^{\bar{3}}\big]|0,0,0\rangle = \bigoplus_{i,j,k=0}^{\infty} \mathbb{C}|i,j,k\rangle$$

- Covariant coordinates:

$$X^i = x^i + \mathrm{i}\,\theta^{ij}\,A_j\ , \qquad Z^a = \tfrac{1}{\sqrt{2}}\big(X^{2a-1} + \mathrm{i}\,X^{2a}\big)\ (a = 1,2,3)$$

- Instanton equations become algebraic equations:

$$\big[Z^a\,, Z^b\big] = 0\ , \qquad \big[Z^a\,, \bar{Z}^{\bar{a}}\,\big] = 3$$

- Vacuum $F_A = 0$ given by harmonic oscillator algebra: $Z^a = z^a$

Noncommutative instantons

- For general solution, fix $n \geq 1$ and let U_n be a partial isometry on $\mathcal{H}$ projecting out all states $|i,j,k\rangle$ with $i+j+k<n$:

$$U_n^\dagger \, U_n \;=\; 1-\Pi_n \ , \quad U_n \, U_n^\dagger \;=\; 1 \ , \quad \Pi_n \;=\; \sum_{i+j+k<n} |i,j,k\rangle\langle i,j,k|$$

- Ansatz: $Z^a \;=\; U_n \, z^a \, f(N) \, U_n^\dagger \ , \qquad N \;=\; \bar{z}^{\bar{a}} \, z^a$

- Topological charge:

$$\ell(n) \;=\; \mathrm{ch}_3 \;=\; -\tfrac{\mathrm{i}}{6} \ \mathrm{Tr}_{\mathcal{H}}(F_A \wedge F_A \wedge F_A) \;=\; \tfrac{1}{6} \, n \, (n+1) \, (n+2)$$

Number of states in $\mathcal{H}$ with $N<n$ (removed by U_n)

Instanton contributions

- U_n identifies full Fock space $\mathcal{H} = \mathbb{C}\big[\bar z^{\bar 1}, \bar z^{\bar 2}, \bar z^{\bar 3}\big]|0,0,0\rangle$ with subspace $\mathcal{H}_{\mathcal{I}} = \bigoplus_{f\in\mathcal{I}} f\big(\bar z^{\bar 1}, \bar z^{\bar 2}, \bar z^{\bar 3}\big)|0,0,0\rangle$:

$$\mathcal{I} = \mathbb{C}\big\langle w_1^i\, w_2^j\, w_3^k \bigm| i+j+k \geq n\big\rangle$$

- Defines plane partition with $|\pi| = \ell(n)$ boxes:

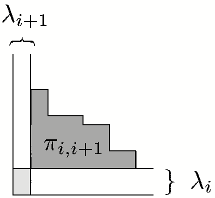
$$\pi = \big\{(i,j,k) \bigm| i,j,k \geq 1\ ,\ w_1^{i-1}\, w_2^{j-1}\, w_3^{k-1} \notin \mathcal{I}\big\}$$

- Instantons sit on top of each other at origin of $\mathbb{C}^3$, and along coordinate axes with asymptotes to $4D$ instantons

- Up to perturbative contributions $\pi = \emptyset$, reproduces MacMahon function $Z_{\mathbb{C}^3} = M(q)$ with $q = \mathrm{e}^{-g_s}$

Melting crystal model in 2D

(Cirafici, Kashani-Poor & RS '09)



- $\{\infty$ Young tableau$\} \longleftrightarrow \mathbb{Z}^2_{\geq 0} \times \{$finite Young tableau$\}$
- Leads to integrable Heisenberg XXZ ferromagnet

  (Dijkgraaf, Orlando & Reffert '09)

Statistical mechanics

- Partition function on bivalent planar graph Γ:

$$Z_{\rm crystal}(X) \;=\; \sum_{\lambda_e} \; \prod_{\text{edges } e} G_{\lambda_e}(q, Q_e) \; \prod_{\substack{\text{vertices} \\ v=(e_1,e_2)}} V_{\lambda_{e_1},\lambda_{e_2}}(q)$$

$$V_{\lambda_{e_1},\lambda_{e_2}}(q) \;=\; \hat{\eta}(q)^{-1}\, q^{-\lambda_{e_1}\,\lambda_{e_2}} \quad , \quad G_{\lambda_e}(q, Q_e) \;=\; q^{a_e\,\frac{\lambda_e\,(\lambda_e-1)}{2}+\lambda_e}\, Q_e^{\lambda_e}$$

- Euler's formula: $\hat{\eta}(q)^{-1} \;=\; \prod_{n=1}^{\infty} \frac{1}{1-q^n} \;=\; \sum_{N=0}^{\infty} p(N)\, q^N$

  $p(N) \;=\;$ number of partitions $\lambda \;=\; (\lambda_1, \lambda_2, \dots)$ ($2D$ Young tableaux) of degree $|\lambda| \;=\; \sum_i \lambda_i \;=\; N$

- Question: Is there a 4D "topological string theory" that reproduces this counting?

$\mathcal{N} = 4$ supersymmetric Yang–Mills theory in 4D

(Vafa & Witten '94)

- $\mathcal{N} = 4$ Vafa–Witten topologically twisted $U(1)$ Yang–Mills on Kähler 4-manifold X, with instanton and monopole charges:

$$n = \frac{1}{8\pi^2} \int_X F_A \wedge F_A , \qquad u_i = \frac{1}{2\pi} \int_{S_i} F_A$$

- Path integral: $Z_{\rm gauge}(X) = \sum_{n,u_i} \Omega(n,u_i)\, q^n \prod_{i=1}^{b_2(X)} Q_i^{u_i}$

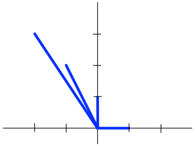
  $\Omega(n,u_i) =$ Witten index $\equiv$ Euler character of moduli space of $U(1)$ instantons on X (anti-self-duality $\star F_A = -F_A$)

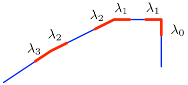
- Conjectural exact expression on Hirzebruch–Jung spaces

  (Fucito, Morales & Poghossian '06; Griguolo *et al.* '07)

Example — ALE spaces

- Resolution of A_n singularity $\mathbb{C}^2/\mathbb{Z}_{n+1}$:





- Melting crystal: $Z_{\mathrm{crystal}}(A_1) \;=\; \dfrac{1}{\hat{\eta}(q)^2} \displaystyle\sum_{\lambda=0}^{\infty} q^{\lambda^2}\, Q^{\lambda}$

- Gauge theory: $Z_{\mathrm{gauge}}(A_1) \;=\; \dfrac{1}{\hat{\eta}(q)^2} \displaystyle\sum_{u=-\infty}^{\infty} q^{-\frac{1}{4}\,u^2}\, Q^{u}$

- Problems related but not identical in 4D!