SUSY, FROM QM TO ANYONS

PAH, with

L. Feher, L. O'Raifeartaigh, F. Bloore

(1988-91)

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(1993-94)

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(2004-2010)

Yerevan, 24-28 Aug. 2010

SUSY and SELF-DUALITY

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- “On Schrödinger superalgebras,”
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- "Bosonized supersymmetry of anyons and supersymmetric exotic particle on the non-commutative plane"
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- "Bosons, fermions and anyons in the plane, and supersymmetry"
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- "Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons,"
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D'Hoker, Vinet PRL **55**, 1043 (1985) "dyon" :

$$H_1 = \underbrace{\left(-\frac{\boldsymbol{\pi}^2}{2} + \frac{q^2}{2}\left(1 - \frac{1}{r}\right)^2\right)}_{H_0} \mathbb{1}_2 - q\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r^3}, \qquad (1)$$

$\boldsymbol{\pi} = -i\boldsymbol{D}$; $\boldsymbol{D} = \boldsymbol{\nabla} - ie\boldsymbol{A}$, $\boldsymbol{\nabla}\times e\boldsymbol{A} = q\frac{\mathbf{r}}{r^3}$. Spin $1/2$ particle with anomalous gyrom. ratio 4 in combined field of Dirac monopole + scalar potential $V = \frac{q^2}{2}\left(1 - \frac{1}{r}\right)^2$. DV find

$$\mathrm{o}(3) \oplus \mathrm{o}(3) \oplus \mathrm{o}(3) \qquad (2)$$

dynamical symmetry, generated by

$$\boldsymbol{J} = \underbrace{\mathbf{r}\times\boldsymbol{\pi} - q\frac{\mathbf{r}}{r}}_{\boldsymbol{L}_0} + \frac{\boldsymbol{\sigma}}{2} \quad \text{ang mom} \qquad (3)$$

+ "Runge-Lenz" vector $\boldsymbol{K}^{DV}$

+ "spin-like" vector $\boldsymbol{\Omega}$.

Bound states for $E < q^2/2$. H-type spectrum

$$E_p = \frac{q^2}{2}\left(1 - \left(\frac{q}{p}\right)^2\right), \quad p = |q|, |q| + 1, \ldots. \qquad (4)$$

$p > |q|$ i.e. have degeneracy $2(p^2 - q^2)$.

$p = |q|$ i.e. $E = 0$ ground state has degeneracy $2|q|$.

• ORIGIN OF DYNAMICAL SYMMETRY ?

McIntosh-Cisneros, Zwanziger 1968 : scalar Hamiltonian

$$H_0 = \frac{\boldsymbol{\pi}^2}{2} + \frac{q^2}{2}\left(1 - \frac{1}{r}\right)^2 \qquad (5)$$

$\boldsymbol{\pi} = -i\boldsymbol{D}$; $\boldsymbol{D} = \boldsymbol{\nabla} - ie\boldsymbol{A}, \boldsymbol{\nabla}\times e\boldsymbol{A} = q\frac{\mathbf{r}}{r^3}$ monopole. q (half)integer. Fine-tuned inverse-square term $\boxed{q^2/2r^2}$ cancels effect of monopole $\rightsquigarrow$

$$H_0 = -\frac{1}{2}\left(\partial_r + \frac{1}{r}\right)^2 + \frac{\boldsymbol{L}_0^2}{2r^2} + \frac{q^2}{2} - \frac{q^2}{r} \qquad (6)$$

where $\boldsymbol{L}_0 = \mathbf{r}\times\boldsymbol{\pi} - q\hat{\mathbf{r}}$ conserved orbital angular momentum $\rightsquigarrow$ H-atom-type spectrum

$$E_p = \frac{q^2}{2}\left(1 - \frac{q^2}{p^2}\right), \qquad p = |q| + 1, \ldots. \qquad (7)$$

Degeneracy half of that of D-V,

$$p^2 - q^2$$

[$= (p + |q|)(p - |q|)$ integer !]

Kepler-type dynamical symmetry

$$\begin{aligned} \boldsymbol{L}_0 &= \mathbf{r} \times \boldsymbol{\pi} - q\,\hat{\mathbf{r}} && \text{ang mom} \\ \boldsymbol{K}_0 &= \tfrac{1}{2}\big(\boldsymbol{\pi} \times \boldsymbol{L}_0 - \boldsymbol{L}_0 \times \boldsymbol{\pi}\big) - q^2\,\hat{\mathbf{r}} \\ & && \text{Runge-Lenz} \end{aligned} \tag{8}$$

Dividing $\boldsymbol{K}_0$ by $\sqrt{(q^2 - 2H)}$ yields

$$\mathrm{o}(4) \qquad \mathrm{e}(3) \qquad \mathrm{o}(3,1) \tag{9}$$

dynamical symmetry, depending on energy being smaller/equal/larger as $q^2/2$.

- For bound motions

$$\mathrm{o}(4) \approx \mathrm{o}(3) \oplus \mathrm{o}(3)$$

$\Rightarrow$ spectrum (7) from representation theory.

- For scattered motions $\mathrm{o}(3,1)$ S-matrix Zwanziger

• PHYSICAL INTERPRETATION ?

NR limit of Dirac eqn. $\rightsquigarrow$ Pauli eqn. for spin 1/2 part in Yang-Mills-Higgs field : $D_i = \partial_i - ie[A_i, \cdot]$,

$$i\partial_t \psi = H\psi \equiv \tag{10}$$

$$\frac{1}{2m}\Big(\boldsymbol{\pi}^2 + (\Phi^a T_a)^2 + \boldsymbol{\sigma}\cdot(\boldsymbol{B}^a + \boldsymbol{D}\Phi^a + A_0^a)T_a\Big)\psi$$

SD 't Hooft-Polyakov monopoles

$SU(2)$ monopole $(\Phi, \boldsymbol{A})$: static, magnetic ($A_0 = 0$) solution of Bogomolny eqn

$$\boldsymbol{D}\Phi = \boldsymbol{B} \ (B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}). \tag{11}$$

Finite-energy condition $|\Phi| \to 1, |\mathbf{r}| \to \infty \rightsquigarrow$ asymptotic Higgs defines mapping $\mathbb{S}^2_\infty \to \mathbb{S}^2$. Winding number $m = [\Phi] \in \pi_2(\mathbb{S}^2) \approx \mathbb{Z} \equiv$ topological charge.

Putting $A_4 = \Phi$, static, magnetic $3D$ YMH $(\Phi, \boldsymbol{A})$ can be viewed as pure YM (A_μ) in $\mathbb{R}^4$. Bogomolny eqn (11) becomes self-duality

$$F_{ij} = \frac{1}{2}\epsilon_{ijkl}F^{kl}. \tag{12}$$

Chiral SUSY of Dirac operator

Dirac operator [cf. Wipf,Ivanov ...]

$$\not{D} = \gamma^\mu D_\mu, \tag{13}$$

where γ^μ Dirac matrices on $\mathbb{R}^4$.

$$\not{D} = -\frac{i}{\sqrt{2}}\begin{pmatrix} 0 & Q^\dagger \\ Q & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Phi \mathbb{1}_2 - i\boldsymbol{\sigma}\cdot\boldsymbol{\pi} \\ \Phi \mathbb{1}_2 + i\boldsymbol{\sigma}\cdot\boldsymbol{\pi} & 0 \end{pmatrix}$$

$$-\frac{1}{2}\not{D}^2 = \begin{pmatrix} Q^\dagger Q & 0 \\ 0 & QQ^\dagger \end{pmatrix}. \tag{14}$$

Straightforward:

$$\begin{aligned} 2Q^\dagger Q &= \Phi^2 - \boldsymbol{D}^2 - \boxed{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\Phi + \boldsymbol{B})} \\ 2QQ^\dagger &= \Phi^2 - \boldsymbol{D}^2 - \boxed{\boldsymbol{\sigma}\cdot(\boldsymbol{D}\Phi - \boldsymbol{B})} \end{aligned}$$

For SD monopole $\boldsymbol{B} = \boldsymbol{D}\Phi$, thus

$$\begin{aligned} Q^\dagger Q &= \tfrac{1}{2}(\Phi^2 - \boldsymbol{D}^2) - \boldsymbol{\sigma}\cdot\boldsymbol{B} &= H_1 \\ QQ^\dagger &= \tfrac{1}{2}(\Phi^2 - \boldsymbol{D}^2) &= H_0. \end{aligned} \tag{15}$$

Hamiltonian H=

$$-\frac{1}{2}\not{D}^2 = \begin{pmatrix} Q^\dagger Q & 0 \\ 0 & QQ^\dagger \end{pmatrix} = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}. \tag{16}$$

H_1, H_0 describe spin $\frac{1}{2}$ particles with anomalous gyromagnetic ratios $g = 4$ and $g = 0$.

Total Hilbert space decomposed into "upper" and "lower" components,

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0 \tag{17}$$

eigensectors of chirality operator

$$\Gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ \Gamma\Big|_{\mathcal{H}_1} = -\mathbb{1}_2, \ \Gamma\Big|_{\mathcal{H}_0} = \mathbb{1}_2 \tag{18}$$

Unitary operator

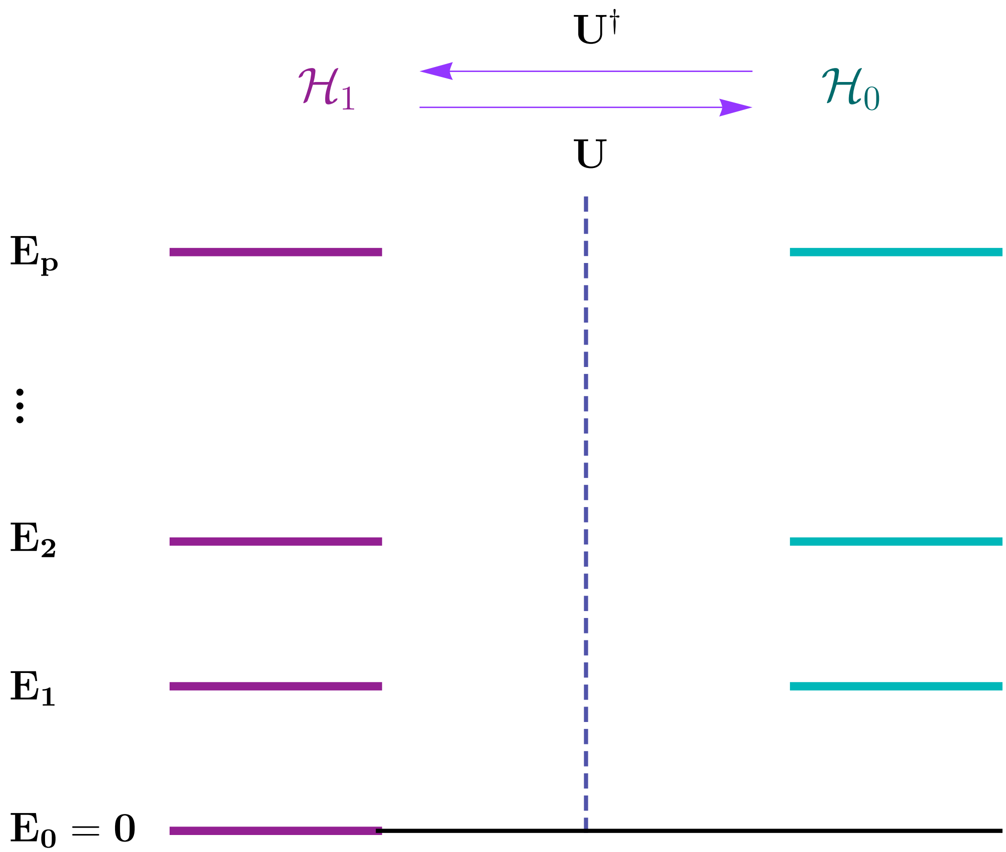
$$U = Q\frac{1}{\sqrt{H_1}} \quad : \mathcal{H}_1 \to \mathcal{H}_0, \tag{19}$$

$$U^\dagger = U^{-1} = \frac{1}{\sqrt{H_1}}Q^\dagger \quad : \mathcal{H}_0 \to \mathcal{H}_1 \tag{20}$$

$$U^\dagger H_0 U = \left(\frac{1}{\sqrt{Q^\dagger Q}}Q^\dagger\right)\left(QQ^\dagger\right)\left(Q\frac{1}{\sqrt{Q^\dagger Q}}\right) = Q^\dagger Q = H_1,$$

i.e., U, $U^\dagger$ intertwine H_0 and H_1,

$$U^\dagger H_0 U = H_1, \quad U H_1 U^\dagger = H_0. \tag{21}$$



The Hilbert space of a SUSY QM system splits into the direct sum of two supbspaces. The energy levels come in pairs, intertwined by an unitary transformation U. One of the sectors may have a zero-energy ground state. Its multiplicity is counted by Atiyah-Singer index theorem.

If $\psi^{(1)}$ is eigenvector of H_1,

$$H_1\psi^{(1)} = E\psi^{(1)},$$

then, for $\psi^{(0)} = Q\psi^{(1)}$,

$$H_0\psi^{(0)} = (QQ^\dagger)(Q\psi^{(1)}) =$$

$$Q\underbrace{(Q^\dagger Q)}_{H_1}(\psi^{(1)}) = E(Q\psi^{(1)}) = E\psi^{(0)},$$

i.e., $\psi^{(0)} = Q\psi^{(1)}$ is eigenvector of H_0 with same eigenvalue E, *provided* $\mathbf{Q}\psi^{(1)} \neq \mathbf{0}$.

Nonzero energy levels come in pairs.

$$Q\psi_0^{(1)} = 0 \quad \Leftrightarrow \quad \not{D}\begin{pmatrix} \psi_0^{(1)} \\ 0 \end{pmatrix} = 0. \qquad (22)$$

solutions of (22) is Atiyah-Singer (AS) index.

<u>THM</u> AS index only depends on *topology* and not on gauge field. For SD monopole & spin 1/2, AS index is em, topological charge.

Exporting symmetries

Let K_0 constant of motion for H_0 dynamics, $[H_0, K_0] = 0$.

$$[H_1, Q^\dagger K_0 Q] = [Q^\dagger Q, Q^\dagger K_0 Q] =$$
$$Q^\dagger \Big([QQ\dagger, K_0]\Big) Q = Q^\dagger \Big([H_0, K_0]\Big) Q = 0$$

i.e.

$$K_1 = U^\dagger K_0 U \tag{23}$$

conserved for H_1 : $[H_1, K_1] = 0$.

Example: "lower component" of D'H-V, $H_0 \mathbb{1}_2$, has $g = 0 \rightsquigarrow$ spin uncoupled :

$$\boldsymbol{S}_0 = \frac{1}{2}\boldsymbol{\sigma} \tag{24}$$

trivially conserved.

$$\begin{aligned} \boldsymbol{S}_1 \;&=\; U^\dagger \boldsymbol{S}_0 U = \\ &-\frac{1}{2H_1}\underbrace{\Big(\frac{1}{2}(\boldsymbol{\pi}^2 - \Phi^2)\boldsymbol{\sigma} + \Phi(\boldsymbol{\pi}\times\boldsymbol{\sigma}) - (\boldsymbol{\sigma}\cdot\boldsymbol{\pi})\boldsymbol{\pi}\Big)}_{\boldsymbol{\Omega}} \end{aligned} \tag{25}$$

conserved for H_1. ($\boldsymbol{\Omega}$ is that of D-V.)

N.B. mixing bosonic and supersymmetries yields superalgebras.

Bogomolny-Prasad-Sommerfeld case

BPS monopole "hedgehog"

$$\Phi^a = -\frac{x^a}{r}\left(\coth r - \frac{1}{r}\right), \; A_i^a = \epsilon_{aik}\frac{x^a}{r^2}\left(1 - \frac{r}{\sinh r}\right).$$

$m = 1$. Manifest spherical symmetry $\rightsquigarrow$ conserved total angular momentum, $\boldsymbol{J}$. $U\boldsymbol{J}U^\dagger = \boldsymbol{J}$ invariant.

For large distances:

$$\Phi^a \to -\frac{x^a}{r}\left(1 - \frac{1}{r}\right), \qquad A_i^a \to \epsilon_{aik}\frac{x^a}{r^2}.$$

Electric charge operator

$$Q_{em} = -\frac{x^a}{r}T_a, \tag{26}$$

where T_a generates su(2). Electric charge

$$Q_{em}\Psi = q\Psi, \quad [Q_{em}, H^\infty] = 0 \tag{27}$$

electric charge asymptotically conserved.

$$\boxed{H_1 \to H_1^{DV}, \quad H_0 \to H_0^{MICZ}\mathbb{1}_2} \tag{28}$$

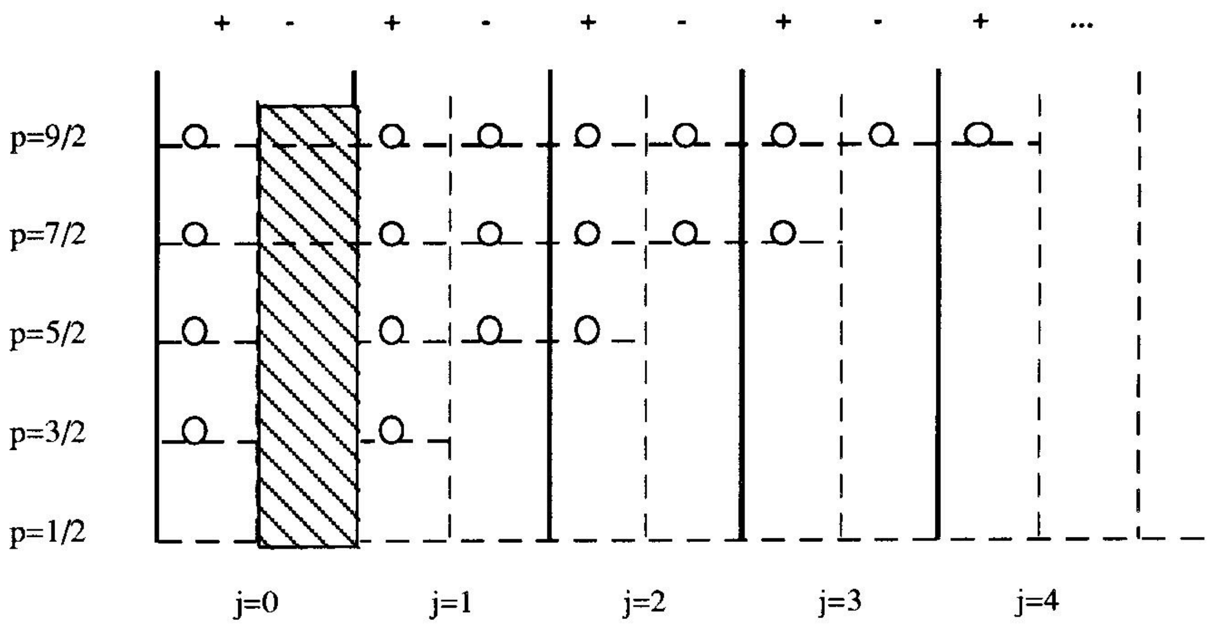
In this limit MICZ Runge-Lenz $\boldsymbol{K}_0$ conserved.
$\boldsymbol{K}_1 = U^\dagger(\boldsymbol{K}_0 \mathbb{1}_2)U$, i.e., $\boldsymbol{K}_1 =$

$$\boldsymbol{K}_0 \mathbb{1}_2 + \boldsymbol{\pi} \times \boldsymbol{\sigma} + \left(\frac{q}{r} - \frac{q}{2}\right)\boldsymbol{\sigma} - (\boldsymbol{\sigma} \cdot \boldsymbol{B})\mathbf{r} - q\frac{\boldsymbol{\Omega}}{2H_1}$$

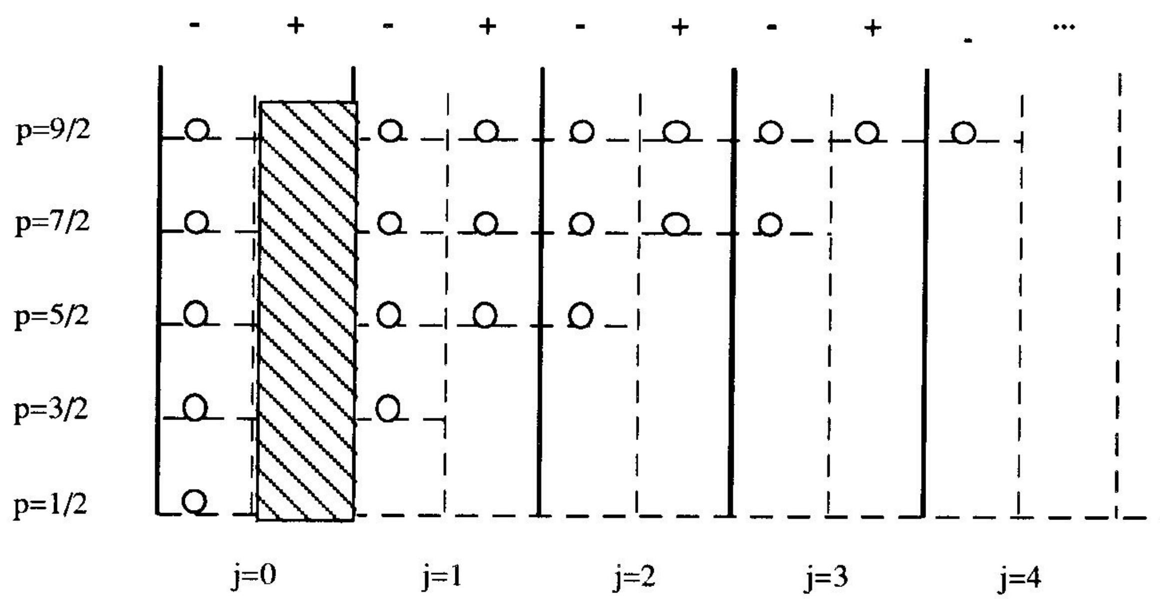
conserved for H_1.

N.B. D-V find $\boldsymbol{K}^{DV} = \boldsymbol{K}_1 + q\dfrac{\boldsymbol{\Omega}}{2H_1}$.

All 3 conserved vectors of D-V recovered by "export



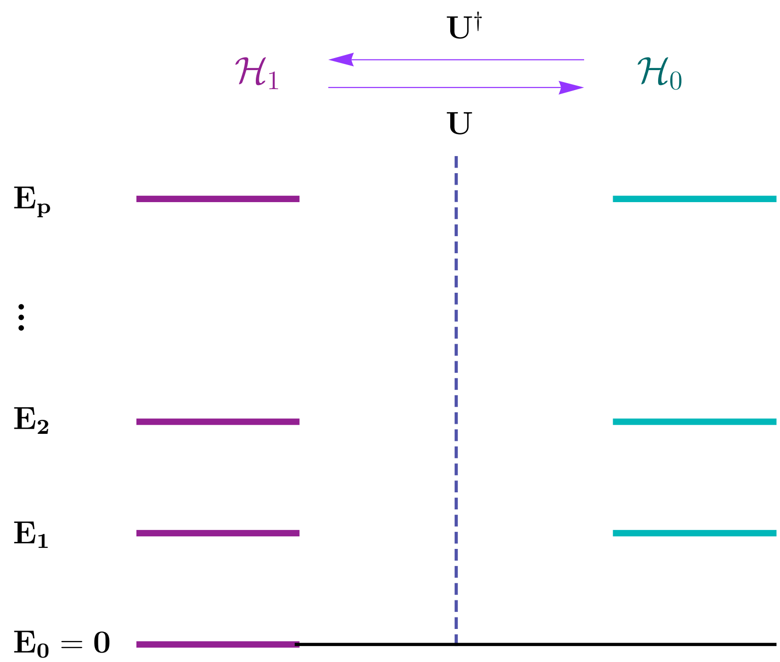
Spectrum of H_0 ($q = 1/2$). No $p = 1/2$ state for $j = 0$.



Spectrum of H_1 ($q = 1/2$). For $p = 1/2$, $j = 0$ one has zero-energy ground state.

GENERAL PHILOSOPHY

SUSY ALLOWS TO DESCRIBE
"COMPLICATED" SYSTEMS
BY "EXPORTING" SIMPLE ONES



SYMMETRIES OF CAN ALSO BE
"EXPORTED" TO "COMPLICATED"
SECTOR.

BOSONIC SYMMETRIES MIX WITH
SUPERSYMMETRIES TO YIELD
SUPERALGEBRAS

N.B. may or may *not* be related to BOSONS
& FERMIONS

Dirac & **DJT** particles in plane

Massive relativistic spinning particles in the plane :

- Dirac: $\boxed{\text{spin } 1/2}$

$$\mathcal{D}_a{}^b\,\psi_b \equiv (P_\mu\gamma^\mu - m\mathbb{1})_a{}^b\psi_b = 0 \qquad (29)$$

where $\psi = (\psi_a)$ 2-component Dirac spinor. Planar Dirac [Pauli] matrices

$$J_0^- = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_1^- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J_2^- = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

generate spin 1/2 representation of planar Lorentz group.

- topologically massive (Deser-Jackiw-Templeton : $\boxed{\text{spin } 1}$

$$\mathfrak{D}_\mu{}^\nu F_\nu \equiv \left(-i\epsilon_{\mu\lambda}{}^\nu P^\lambda + m\delta_\mu{}^\nu\right) F_\nu = 0\,, \qquad (30)$$

(F_μ) 3 component vector. $-i\epsilon_{\mu\lambda}{}^\nu$, i.e.

$$J_0^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad J_1^+ = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad J_2^+ = \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generate 3-D vector representation.

SUSY unification

Direct sum $D^{1/2} \oplus D^1$, Lorentz generators

$$\mathcal{J}_\mu = \left(\begin{array}{c|c} J_\mu^- & 0 \\ -- & -- \\ 0 & J_\mu^+ \end{array} \right). \tag{31}$$

Unified wave function 5-tuplet

$$\Psi = \left(\begin{array}{c} \psi_a \\ -- \\ F_\mu \end{array} \right), \; (\psi_a) = \left(\begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \; (F_\mu) = \left(\begin{array}{c} F_0 \\ F_1 \\ F_2 \end{array} \right).$$

Adding off-diagonal matrices

$$L_1 = \sqrt{2} \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & i \\ 0 & 0 & 1 & 0 & 0 \\ - & - & - & - & - \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \end{array} \right),$$

$$L_2 = \sqrt{2} \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i \\ - & - & - & - & - \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \end{array} \right).$$

Completes Lorentz algebra generated by $\mathcal{J}_\mu$ to $\mathfrak{osp}(1|2)$ superalgebra

$$[\mathcal{J}_\mu, \mathcal{J}_\nu] = -i\epsilon_{\mu\nu\lambda}\mathcal{J}^\lambda\,,$$
$$\{L_A, L_B\} = 4(\mathcal{J}\gamma)_{AB}\,, \qquad (32)$$
$$[\mathcal{J}_\mu, L_A] = \tfrac{1}{2}(\gamma_\mu)_A{}^B L_B\,,$$

where $(\gamma^\mu)_{AB} = \epsilon_{BC}(\gamma^\mu)_A{}^C\,.$

Sectors distinguished by reflection operator

$$R = \mathrm{diag}\Big(-\mathbb{1}_2, \mathbb{1}_3\Big), \qquad (33)$$

$R^2 \;=\; \mathbb{1},\, \{L_A, R\} = 0\,.$

Super-Casimir operator

$$\mathcal{C} = \mathcal{J}_\mu\mathcal{J}^\mu - \frac{1}{8}L^A L_A = -\frac{3}{2} \qquad (34)$$

$\Rightarrow$ $\mathfrak{osp}(1|2)$ representation is irreducible.

Original ingredients, (ψ_a) and (F_μ), recovered by projecting onto ∓ 1 eigenspaces of reflection operator, R. On these subspaces Casimir of Lorentz subalgebra is

$$\mathcal{J}_\mu \mathcal{J}^\mu = -\widehat{\alpha}(\widehat{\alpha} - 1)\,, \quad \widehat{\alpha} = -\frac{1}{4}(3 + R)\,. \qquad (35)$$

$\widehat{\alpha}$ has eigenvalues $\alpha_- = -\frac{1}{2}$ and $\alpha_+ = -1$ $\Rightarrow$

Eigenspaces carry irreducible spin-$1/2$ (Dirac) and spin-1 DJT representations, respectively.

Dirac & DJT can be written in same form,

$$(P_\mu \mathcal{J}^\mu - \hat{\alpha} m)\Psi = 0. \tag{36}$$

Operators L_A interchange ψ and F, but do not preserve physical states $\equiv$ solutions of the Dirac and DJT eqns. Can be achieved by considering instead the two supercharges

$$\mathcal{Q}_1 = \frac{1}{2\sqrt{m}}\Big(L_2 P_+ + L_1(mR - P_0)\Big), \tag{37}$$

$$\mathcal{Q}_2 = \frac{1}{2\sqrt{m}}(-L_1 P_- + L_2(mR + P_0)), \tag{38}$$

where $P_\pm = P_1 \pm iP_2$. Action on spin-1 (F_μ) and spin-1/2 (ψ_a) components :

$$\Psi' = \begin{pmatrix} \psi'_a \\ F'_\mu \end{pmatrix} = \zeta^A \begin{pmatrix} \mathcal{Q}_{Aa}{}^\mu F_\mu \\ \mathcal{Q}_{A\mu}{}^a \psi_a \end{pmatrix}, \tag{39}$$

where ζ^A parameters of SUSY transformation. Two-component Dirac field ψ_a transformed into three-component DJT field F'_μ and conversely.

$\boxed{\text{SUSY interchages sectors}}$

Furthermore,

$$\boxed{\mathcal{D}_a{}^b\,\psi'_b} = \qquad (40)$$

$$\zeta^A\left(\mathcal{Q}_{Aa}{}^\mu\boxed{\mathfrak{D}_\mu{}^\nu F_\nu} + \frac{1}{2\sqrt{m}}Q_{Aa}{}^\mu(P^2+m^2)F_\mu\right),$$

$$\boxed{\mathfrak{D}_\mu{}^\nu F'_\nu} = \qquad (41)$$

$$\zeta^A\left(-\frac{1}{2}\mathcal{Q}_{A\mu}{}^a\boxed{\mathcal{D}_a{}^b\psi_b} - \frac{1}{2\sqrt{m}}Q_{A\mu}{}^a(P^2+m^2)\psi_a\right).$$

Both Dirac & DJT eqns imply Klein-Gordon eqn $(P^2+m^2)=0 \Rightarrow$ transformed fields satisfy on shell Dirac & DJT eqns if original ones satisfy them in reversed order.

N.B. NR limit $\rightsquigarrow$ centrally extended Super-Schrödinger symmetry.

ANYONS

Anyons correspond to irreducible representations of planar Poincaré group. Characterized by 2 Casimir invariants

$$\begin{aligned} &\mathbf{P}^2 + m^2c^2 = 0 \quad \text{mass-shell} \\ &P_\mu J^\mu - scm = 0 \quad \text{Pauli-Lubanski} \end{aligned} \tag{42}$$

Jackiw, Nair 1990 combine spin 1 representation carried by topologically massive (TM) vector system with fractional spin, carried by half-bounded representation. JN wave fct is

$$F_\mu(z, x) = \sum_n f_n(z) F_\mu^n(x) \ , \tag{43}$$

where $f_n = z^n$, $n = 0, 1, 2, \ldots$, infinite dimensional orthonormal basis in internal space. $F_\mu^n(x)$ is, for each internal index n, a (TM) wave function.

cf. also Nersessian 1997

JN describe anyons by **Pauli-Lubanski** eqn.

$$\boxed{(P^{\mu}\mathfrak{J}^{+}_{\mu} - \beta_{+} m)F = 0\,,\ \beta_{+} = \alpha - 1,} \tag{44}$$

$F = (F_\mu)$. $\mathfrak{J}^{+}_{\mu}$ generates direct sum of planar Lorentz algebras,

$$\mathfrak{J}^{+}_{\mu} = J^{+}_{\mu} + j_{\mu}, \tag{45}$$

where $(J^{+}_{\mu})_{\nu}{}^{\lambda} = i\epsilon_{\mu\nu}{}^{\lambda}$ generates TM, spin 1 repr of Lorentz algebra. j_μ, carrying fractional spin, belongs to discrete series D^{+}_{α}. J^{+}_{μ} acts on vector index of F_μ, j_μ acts on internal (fractional) part, labeled by n.

(44) only fixes one Casimir of planar Poincaré $\rightsquigarrow$ supplemented by subsidiary conditions. Those of JN equivalent to

$$\boxed{P^{\mu}F_{\mu} = 0\,, \quad \epsilon^{\mu\nu\lambda}P_{\mu}\, j_{\nu}\, F_{\lambda} = 0\,.} \tag{46}$$

Plyushchay 1991 (44)+(46) imply **TM** + 2+1D Majorana-type eqns [seen before for particle with torsion]

$$\mathfrak{D}_\mu{}^\nu F_\nu \equiv \left(-i\epsilon_{\mu\lambda}{}^\nu P^\lambda + m\delta_\mu{}^\nu\right) F_\nu = 0 \qquad (47)$$

$$(P_\mu j^\mu - \alpha m)F = 0\,. \qquad (48)$$

Plyushchay 91 : slightly different approach. Wave fcts

$$\psi_a(x,z) = \sum_n f_n(z)\,\psi_a^n(x)\,, \qquad (49)$$

where $\psi_a^n(x)$ 2-component "Dirac" [Pauli] spinor. Posits **Dirac** + **Majorana** eqns

$$\mathcal{D}_a{}^b\,\psi_b \equiv (P_\mu\gamma^\mu - m)_a{}^b\psi_b = 0\,, \qquad (50)$$

$$(P_\mu j^\mu - \alpha m)\psi = 0\,. \qquad (51)$$

Eqns (50)-(48) imply

$$\boxed{(P^\mu \mathfrak{J}_\mu^- - \beta_- m)\psi = 0\,, \quad \beta_- = \alpha - \tfrac{1}{2}\,,} \qquad (52)$$

where

$$\mathfrak{J}_\mu^- = J_\mu^- + j_\mu, \qquad (53)$$

$$(J_\mu^-)_a{}^b = -\tfrac{1}{2}(\gamma_\mu)_a{}^b \qquad (54)$$

generate spin 1/2 rep. of planar Lorentz group.

Relation of two approaches

THM : Jackiw-Nair and Dirac-Majorana approaches are two facets of the same supersymmetric system.

Proof : both described by eqns of same form,

$$D^{\pm}\psi^{\pm} = 0, \tag{55}$$

$$(P^{\mu}j_{\mu} - \alpha m)\psi^{\pm} = 0, \tag{56}$$

where

$$\begin{aligned} D^{+} &= \mathfrak{D} && \text{TM} \\ D^{-} &= \mathcal{D} && \text{Dirac} \end{aligned} \tag{57}$$

cf. eqrefDJT and (50), resp, and put $\psi^{-} = \psi$ and $\psi^{+} = F$.

N.B. : posited first-order eqns imply Klein-Gordon eqn.

ψ^- and ψ^+ fields have fractional spins

$$\beta_- = \alpha - \frac{1}{2}, \qquad \beta_+ = \alpha - 1 \tag{58}$$

shifted by

$$\beta_- - \beta_+ = 1/2, \tag{59}$$

& have same masses. Can be unified into supermultiplet along same lines as for TM/Dirac :

$$(P_\mu \mathcal{J}^\mu - \hat{\alpha} m)\Psi = 0, \tag{60}$$

$$(P_\mu j^\mu - \alpha m)\Psi = 0, \tag{61}$$

$$\mathcal{J}^\mu = diag(\mathfrak{I}_\mu^-, \mathfrak{I}_\mu^+). \tag{62}$$

Ψ obtained by putting together DM and JN fields, $\Psi = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix}$, and $\hat{\alpha}$ is diagonal operator $diag(\alpha_-, \alpha_+)$,

$$\alpha_- = -\tfrac{1}{2} \quad \text{for} \quad \Psi = \begin{pmatrix} \psi^- \\ 0 \end{pmatrix},$$

$$\alpha_+ = -1 \quad \text{for} \quad \Psi = \begin{pmatrix} 0 \\ \psi^+ \end{pmatrix}.$$

(60) is SUSY eqn which unifies Dirac & TM, and is supplemented by Majorana eqn (61).